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The Group-Membership of Singular Matrices.

BY ARTHUR RANUM.

INTRODUCTION.

It is well-known* that the totality of non-singular matrices (linear homogeneous substitutions with non-vanishing determinants) form a group under the usual law of composition or multiplication. It has not been so generally recognized that there exist sets of singular matrices (linear substitutions with vanishing determinants), which also form groups† under the same law of composition. For instance, the set of binary matrices of the form $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$, $a \neq 0$, obviously form a group of which $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the identical matrix.

Not all singular matrices, however, belong to groups; *e. g.*, $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ does not belong to any group whatsoever. It becomes of interest, therefore, to determine whether a given matrix belongs to any group or not; if it does, to find the nature of the largest group to which it belongs; and if it does not belong to any group, to show that there always exists a certain simple relation between it and a uniquely determinable group.

In short, we wish to make a complete classification of all n-ary matrices as to their group-membership and exact relationship to groups. It will also be found that we shall naturally be led to certain incidental results on the roots of a matrix.

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* *Weber's Algebra*, Vol. II, 2nd edition (1899), § 41, p. 168.

† In this paper the word *group* will be used in the generally accepted sense and not in the broader sense recently given to it by Frobenius and Schur in the *Berliner Sitzungsberichte*, 1906, I, p. 209, and by Autonne in the *Comptes Rendus*, 1906, Vol. CXLIII, p. 670. Frobenius and Schur in their other papers employ the word in its usual sense.

PART I. GROUP-MEMBERS.

Definitions and Preliminary Notions.

1. Throughout this paper the elements of the matrices will be unrestricted real or complex numbers, belonging, therefore, to the continuous domain. The number of rows or columns of a matrix will be called its *degree*. As usual, a matrix will be said to be of *rank* r, if it contains at least one r-rowed minor determinant that does not vanish, while all its $r+1$-rowed minor determinants vanish. Following Sylvester, the *vacuity*[*] of a matrix will be defined as the number of zero roots of its characteristic equation. A non-singular matrix of degree n is therefore of vacuity 0 and of rank n, while a singular matrix is of vacuity $v > 0$ and of rank $r < n$. The matrix zero (whose elements are all zero) will be denoted by the symbol 0_n, where n is its degree. A matrix different from zero is said to be *nilpotent*, if one of its powers is equal to zero. The nilpotent matrices are known to be precisely those whose vacuity is equal to their degree and whose rank is > 0.[†] The unit matrix of degree n will be denoted by the symbol U_n.

Two matrices M and M' are said to be *similar*, if there exists a non-singular matrix L such that $L^{-1}ML = M'$. L is said to *transform* M into M'. In the same way two sets of matrices S and S' are similar, if there exists a non-singular matrix that transforms all the matrices of S into those of S'. A set of matrices S is said to be *completely reducible*, or simply *reducible*,[‡] if there exists a similar set S', all of whose matrices are of the form

$$\begin{pmatrix} M_a & 0 \\ 0 & M_b \end{pmatrix},$$

where the symbols M_a and M_b stand for *component* matrices of degrees a and b respectively, and the elements of the remaining rectangular matrices are all zero. Symbolically, we may write

$$S' = \begin{pmatrix} S'_a & 0 \\ 0 & S'_b \end{pmatrix}.$$

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[*] AMERICAN JOURNAL OF MATHEMATICS, Vol. VI (1884), p. 278.

[†] Frobenius, *Crelle's Journal*, Vol. LXXXIV (1878), p. 15, VI.

[‡] It is well-known that there are groups of non-singular matrices that are reducible to the form

$$\begin{pmatrix} G_{aa} & 0 \\ G_{ab} & G_{bb} \end{pmatrix}$$

without being completely reducible in the sense defined above. But it will be shown that every group of singular matrices is reducible in the latter sense.

2. From the theory of elementary divisors* we borrow the theorem that every matrix M' of degree n is similar to a matrix of the form

$$M = \begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \quad (u + v = n), \tag{1}$$

in which the component L_u is non-singular and the component N_v is either nilpotent or zero; v is then the vacuity of M'. If M' is non-singular, $v = 0$ and $M = L_n$; on the other hand, if M' is nilpotent or zero, $u = 0$ and $M = N_n$. Let the characteristic equation of M' be written $\Phi(\lambda) = 0$. It has v roots equal to zero. Let the corresponding elementary divisors of the characteristic determinant of M', which we shall call *vacuous* elementary divisors, be denoted by $\lambda^{e_1}, \ldots, \lambda^{e_s}$, where $e_i \geqq e_{i+1}$ $(i = 1, \ldots, s-1)$ and $e_1 + \ldots + e_s = v$. Then $[\overset{\circ}{e}_1 \overset{\circ}{e}_2 \ldots \overset{\circ}{e}_s]$ is the *characteristic*† of the component matrix N_v. Now M can be chosen in such a way that

$$N_v = \begin{pmatrix} E_1 & 0 & \ldots & 0 \\ 0 & E_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & E_s \end{pmatrix}, \tag{2}$$

where

$$E_i = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \text{ of degree } e_i \quad (i = 1, \ldots, s). \tag{3}$$

When this is done, N_v is said to be in its *canonical form*. When the non-singular component L_u is also chosen in its corresponding canonical form, M is said to be the canonical form of M'. Since the irreducible nilpotent component E_i $(i = 1, \ldots, s)$ is obviously of rank $e_i - 1$, it follows that M, and therefore M', is of rank $r = u + \sum\limits_1^s (e_i - 1) = u + v - s = n - s$. If N_v is irreducible, so that $s = 1$, M' is of rank $n - 1$. On the other hand, if any one of the vacuous elementary divisors of the characteristic determinant of M' is linear, *e. g.*, if

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* See Muth's "Elementartheiler" (1899), §§ 77–79, pp. 152–158; also Bôcher's "Introduction to Higher Algebra" (1907), Chap. XXI, and in particular § 100, p. 292. The latter book is especially to be recommended as an introduction both to the theory of matrices and to the theory of elementary divisors.

† See Bôcher's "Higher Algebra," §§ 99 and 100.

$e_s=1$, then $E_s=0$, and M has at least one irreducible zero component. If all of the vacuous elementary divisors are linear, then N_v is zero,

$$M=\begin{pmatrix} L_u & 0 \\ 0 & 0_v \end{pmatrix}, \tag{4}$$

and M' is of rank $r=u=n-v$; in other words, its rank is equal to its degree minus its vacuity.

Groups of Singular Matrices.

3. Consider a set of matrices, finite or infinite in number, of the form

$$M=\begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \quad M'=\begin{pmatrix} L'_r & 0 \\ 0 & 0_v \end{pmatrix}, \text{ etc.,}$$

of rank r and vacuity v, in which L_r, L'_r, etc., are non-singular, if $r>0$.* It is apparent that if $L_r L'_r=L''_r$, then $MM'=M''$, and conversely. Therefore the given set of matrices will form a group $G=\begin{pmatrix} G_r & 0 \\ 0 & 0_v \end{pmatrix}$, if, and only if, their non-singular components form a group G_r. In every such group G the identical matrix is

$$I=\begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix} \tag{5}$$

and the inverse of any matrix M is

$$M^{-1}=\begin{pmatrix} L_r^{-1} & 0 \\ 0 & 0_v \end{pmatrix}.$$

In particular, the totality of the matrices of this form constitutes an r^2-parameter continuous group, of which all the other groups are subgroups. It will be noticed that all its matrices have the same rank r.

We wish to show that every possible group of matrices is similar to some group of the kind just described.

The Rank of a Group.

4. LEMMA: *All the matrices of a group have the same rank.*

For if A, of rank a, and B, of rank b, are any two matrices of a group, then matrices C and C^{-1} can be found to satisfy the equations $A=BC$ and $B=AC^{-1}$; and since the rank of the product of two matrices is equal to, or less than, the rank of either factor, the first of these equations shows that $a\leqq b$ and the second shows that $b\leqq a$; therefore $a=b$.

The rank of the matrices of a group will be called the *rank of the group.*

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* The extreme case, in which $r=0$, can be disposed of at once. The only matrix of rank zero is the matrix zero, which clearly constitutes a group of order one.

5. This lemma compels us to exclude from group-membership every matrix M of the form (1), in which N_v is nilpotent.

For
$$M^{\mu} = \begin{pmatrix} L_u^{\mu} & 0 \\ 0 & N_v^{\mu} \end{pmatrix},$$

and since N_v is nilpotent, μ can be chosen large enough so that $N_v^{\mu} = 0_v$; in that case M^{μ} will be of rank u. Now since N_v is different from 0_v, its rank is > 0 and the rank of M is $> u$; therefore M is of higher rank than M^{μ}. But if M belonged to a group G, M^{μ} would belong to G and, by the lemma, would be of the same rank as M.

The Conditions of Group-Membership.

6. If a matrix belongs to a group, every similar matrix belongs to a similar group, and if a matrix does not belong to a group, no similar matrix can belong to a group. Now since every matrix is similar to a matrix of the form (1), which we have seen to be a group-member, if the component N_v is zero, and a non-group-member, if N_v is nilpotent, therefore we have proved the

THEOREM: *A necessary and sufficient condition for the group-membership of a matrix (of rank r and vacuity v) is its similarity to a matrix of the form*

$$\begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{6}$$

in which L_r is non-singular.

It is apparent from the proof of this theorem that the different powers of a matrix all have the same vacuity, but not necessarily the same rank; they do all have the same rank, if, and only if, the matrix is a group-member.

7. From this theorem, in view of the principles of elementary divisors and of canonical forms as stated in § 2, we easily derive several *characteristic properties of group-members*, any one of which distinguishes them from all non-group-members. Since all non-singular matrices are group-members, it will be convenient to confine the statement of the results to the only significant case, that in which the matrices are singular.

Thus, *a singular matrix belongs to a group, if, and only if, it satisfies any one of the following equivalent conditions:*

(1) *its powers are all of the same rank;*

(2) *its canonical form has no components that are nilpotent;*

(3) *it is reducible to a non-singular matrix bordered with zeros;*

(4) *the vacuous elementary divisors of its characteristic determinant are all linear;*

(5) *its rank is equal to its degree minus its vacuity.*

If the vacuity of a matrix is zero or one, condition (5) is necessarily satisfied. Therefore every matrix of vacuity one, as well as every non-singular matrix, is a group-member. For instance, every ternary matrix whose characteristic equation has at least two roots different from zero is a group-member.

Idempotent Matrices.

8. Now consider the identical matrix I of any group. It satisfies the condition $I^2 = I$ and is therefore *idempotent*; conversely, every idempotent matrix is the identical matrix of some group. Every such matrix, by the theorem of § 6, is similar to a matrix of the form (6), which in this particular case becomes the matrix (5). Accordingly, the roots of its characteristic equation are equal to one or zero, and not only the vacuous elementary divisors of its characteristic determinant, but all the elementary divisors, are linear. Since the latter condition is also sufficient, we have proved the

Theorem: *A matrix I (of rank r and vacuity v) is the identical matrix of a group, if, and only if, it satisfies any one of the following equivalent conditions:*

(1) *it is idempotent, that is, $I^2 = I$;*

(2) *it is similar to a matrix of the form*

$$\begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix}, \tag{7}$$

in which U_r is the unit matrix of degree r;

(3) *the roots of its characteristic equation are equal to one or zero, and the elementary divisors of its characteristic determinant are all linear.*

Its rank r is equal to the multiplicity of the root one, while its vacuity v is equal to the multiplicity of the root zero. If $v = 0$, $I = U_n$, while if $r = 0$, $I = 0_n$; in each of these two extreme cases there is only one identical matrix.

Periodic Matrices.

9. Again, suppose that a matrix M belongs to a group of finite order G; then it may be spoken of as *of finite period*, or simply as *periodic*. If m is its period and I is the identical matrix of G, then $M^m = I$ and $M^{m+1} = M$. Conversely, any matrix that satisfies an equation of the form $M^{m+1} = M$ is periodic; for since the powers of M include an identical matrix M^m and an inverse to M, namely M^{m-1}, they form a group.

If, in the theorem of § 6, the matrix (6) is of period m, then $L_r^m = U_r$;

therefore L_r is periodic in the usual non-singular sense, and can be chosen in the canonical form

$$L_r = \begin{pmatrix} \varepsilon_1 \ldots . 0 \\ \ldots\ldots \\ 0 \ldots . \varepsilon_r \end{pmatrix}, \qquad (8)$$

in which $\varepsilon_1, \ldots, \varepsilon_r$ are mth roots of unity. Thus we have the

THEOREM: *A periodic matrix (of rank r and vacuity v) is characterized by any one of the following equivalent properties:*

(1) *one of its higher powers is equal to its first power;*

(2) *it is similar to a matrix of the form*

$$\begin{pmatrix} L_r & 0 \\ 0 & 0_v \end{pmatrix}, \qquad (6)$$

in which L_r is given by equation (8);

(3) *the roots of its characteristic equation are roots of unity or zero, and the elementary divisors of its characteristic determinant are all linear.*

The Canonical Form of a Group.

10. We now come to the solution of the problem stated in § 3. Let G' be any group whatever, of finite or of infinite order, and let r be its rank, I' its identical matrix, and M' any one of its matrices. Then, by the theorem of § 8, there exists a non-singular matrix L that will transform I' into

$$I = \begin{pmatrix} U_r & 0 \\ 0 & 0_v \end{pmatrix}. \qquad (5)$$

Suppose that L transforms G' into G and M' into

$$M = \begin{pmatrix} L_r & A \\ B & L_v \end{pmatrix},$$

where A and B are rectangular matrices. Since M must satisfy the equations $M = MI = IM$, it follows by direct calculation that $A = 0$, $B = 0$, and $L_v = 0_v$, and that M must be of the form (6); moreover, since its rank is r, its component L_r must be non-singular. But M is obviously any matrix of the group G. Therefore we have arrived at the

THEOREM: *Every group of matrices of rank r is similar to a group*

$$G = \begin{pmatrix} G_r & 0 \\ 0 & 0_v \end{pmatrix}, \qquad (9)$$

in which G_r is a group of non-singular matrices of degree r.[*]

---

[*] For the special case in which the group is of finite order, this theorem was proved by Schur, "Neue Begründung der Theorie der Gruppencharaktere," *Berliner Sitzungsberichte*, 1905, I, p. 418, VIII.

G is said to be a *canonical form* of the group. In other words, a set of matrices will form a group, if, and only if, it is similar to a set of matrices of the form (6), in which the non-singular components L_r themselves form a group.

COROLLARY 1: *Every group of singular matrices is reducible to a group of non-singular matrices, each of which is bordered with zeros.*

COROLLARY 2: *Every group of matrices of rank r is simply isomorphic with a group of non-singular matrices of degree r.*

For instance, every group of rank one is simply isomorphic with a unary non-singular group and is therefore Abelian; again, every group of rank two is simply isomorphic with a binary non-singular group.

Entire Groups.

11. Any group of matrices which is not a subgroup of a larger group will be called an *entire group.* An entire group is thus the largest group G to which any one of its matrices M belongs. Every other group to which M belongs is a subgroup of G. Since no two entire groups can have any matrices in common, it follows that every group-member belongs to one and only one entire group. Group-members can therefore be uniquely classified, first according to their ranks, and second according to the entire groups to which they belong.

Now in view of the theorem of the last section, it is evident that all the entire groups of degree n and rank r are similar to one another and that their canonical form is a continuous r^2-parameter group of the form (9). Their number is obviously infinite, unless $r = n$ or $r = 0$. If $r = n$, there is just one entire group, including the non-singular matrices; if $r = 0$, there is one entire group, including the matrix zero alone.

PART II. NON-GROUP-MEMBERS.

The Group-Index of a Matrix.

12. Let us now consider more in detail the properties of matrices that do not belong to any group. By reference to § 5 we see that if any such matrix be raised to a sufficiently high power, the resulting matrix will surely belong to some group. Thus every matrix whatever has some positive integral power* that belongs to a group; and every non-group-member is a root of some group-member.

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* A negative or zero power of M obviously has no determinate meaning, unless M is a group-member; in the latter case M^0 is the identical matrix of every group to which M belongs and $M^{-\nu}$ is the inverse of M^ν in every such group.

The exponent of the lowest positive integral power to which a matrix must be raised in order to belong to a group will be called the *group-index* of the matrix. Hence the group-index of every group-member is unity. If M is a non-group-member of group-index μ, then M^μ and all the higher powers of M are group-members (and all belong to the same groups), while the lower powers of M are non-group-members. Therefore, if M^μ is a group-member, while $M^{\mu-1}$ is not, then μ is the group-index of M.

13. By reference to the canonical form of a non-group-member as given by equations (1), (2), (3) of §2, we see that if μ is its group-index, then the ranks of its successive powers from the first up to the μth form a continually decreasing series of integers,* while the ranks of its higher powers, from the μth on, are all equal.

Moreover, using the notation of §2, we see that since

$$M^\nu = \begin{pmatrix} L_u^\nu & 0 \\ 0 & N_v^\nu \end{pmatrix},$$

and since N_v^ν is made up of the component matrices E_i^ν $(i = 1, \ldots, s)$, therefore M^ν will be a group-member, if, and only if, $N_v^\nu = 0_v$; that is, if $E_i^\nu = 0_{e_i}$ $(i = 1, \ldots, s)$. Now the lowest value of ν for which the equation $E_i^\nu = 0_{e_i}$ is satisfied is e_i, the degree of E_i. This can be most easily seen by considering a special case. For instance, if $e_i = 4$, then

$$E_i = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad E_i^4 = 0_4,$$

and all the higher powers of E_i are equal to 0_4. Therefore the lowest value of ν for which M^ν is a group-member is the largest one of the integers $e_1, \ldots, e_s$, namely e_1. That is, if μ is the group-index of M, then $\mu = e_1$. This result also holds for singular group-members; for in their case $\mu = e_1 = \ldots = e_s = 1$. Thus we have the general

THEOREM: *The group-index of a singular matrix is equal to the degree of that vacuous elementary divisor of its characteristic determinant which is of highest degree.*

It follows that every non-group-member of group-index μ is similar to a matrix having at least one irreducible nilpotent component of degree μ and none of higher degree; its vacuity must be $\geqq \mu > 1$.

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* This fact was noticed by Taber, AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 370. His statement of it does not involve the notion of group-membership.

Group-Index and Vacuity.

14. Since matrices exist whose characteristic determinants have any prescribed elementary divisors, therefore the degrees $e_1, \ldots, e_s$ of the vacuous elementary divisors can have any positive integral values such that their sum $\sum_1^s e_i = v \leqq n$, where v is the vacuity and n the degree of the matrix. Hence the group-index of a matrix of vacuity v can have any value from one to v inclusive, but can not be greater than v; that is, its maximum value is v. Similarly, the maximum value of the group-index of a matrix of degree n is n.

Accordingly, the vth powers of all matrices of vacuity v are group-members, and the groups to which they belong are of rank $n - v$. The nth powers of all matrices whatever of degree n are group-members.

If the group-index of a non-group-member M is equal to its vacuity v, then there is just one vacuous elementary divisor λ^v $(v > 1)$ and M is similar to a matrix having just one irreducible nilpotent component; and conversely.

15. The extreme case in which $v = n$ is that of the nilpotent matrices and the matrix zero. The group-index μ of a nilpotent matrix is evidently the exponent of the lowest power to which it must be raised in order to give the matrix zero. If μ has its maximum value n, then there is only one elementary divisor λ^n and the matrix is irreducible. Conversely, if a singular matrix of degree n is irreducible, it must be nilpotent and its group-index must have the maximum value n.

Group-Index and Rank.

16. Referring again to §2, we see that every non-group-member of degree n, vacuity v, rank r and group-index μ is similar to a canonical matrix having $n-r$ irreducible nilpotent components, the sum of whose degrees is v, while at least one of them is of degree μ and the degrees of all the rest are $\leqq \mu$; moreover, if $v < n$, the canonical matrix has one non-singular component of degree $n - v$, which may, or may not, be further reducible.

The rank r of a singular matrix M of vacuity v can obviously have any positive integral value from $n - v$ (when M is a group-member) up to $n - 1$ (when its group-index μ has the maximum value v).

If $r = n - v + 1$, then $e_1 = 2$, $e_2 = \ldots = e_s = 1$, and $\mu = 2$. Hence if the rank of a matrix is one greater than the difference between its degree and its vacuity (so that the latter must be > 1), then its group-index is two.

On the other hand, it is clear that if $\mu = v$, $r = n - 1$, and conversely while if $\mu = v - 1$, $r = n - 2$. That is, every matrix whose group-index is equal to its vacuity is of rank one less than its degree, and conversely; while every matrix whose group-index is one less than its vacuity is of rank two less than its degree.

17. As an illustration of the relations existing between the degree n, the vacuity v, the rank r, and the group-index μ, of a matrix, a few of the corresponding values of these four quantities may be tabulated as follows:

$$
\begin{array}{ll}
v = 0, & r = n, \quad \mu = 1 \\
v = 1, & r = n - 1, \quad \mu = 1 \\
v = 2, & \begin{cases} r = n - 2, & \mu = 1 \\ r = n - 1, & \mu = 2 \end{cases} \\
v = 3, & \begin{cases} r = n - 3, & \mu = 1 \\ r = n - 2, & \mu = 2 \\ r = n - 1, & \mu = 3 \end{cases} \\
v = 4, & \begin{cases} r = n - 4, & \mu = 1 \\ r = n - 3, & \mu = 2 \\ r = n - 2, & \mu = 3 \text{ or } 2 \\ r = n - 1, & \mu = 4 \end{cases}
\end{array}
$$

$$
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
$$

$$
v = n, \begin{cases} r = 0, & \mu = 1 \\ r = 1, & \mu = 2 \\ r = 2, & \mu = 3 \text{ or } 2 \\ r = 3, & \mu = 4, 3, \text{ or } 2 \ (\text{if } n \geqq 6) \\ \cdots\cdots\cdots\cdots & \\ r = n - 1, & \mu = n \end{cases}
$$

For the case $v = 4$ the corresponding canonical forms of the nilpotent (or zero) component N_4 are as follows:

$$
r = n - 4, \quad \mu = 1, \quad N_4 = 0_4
$$

$$
r = n - 3, \quad \mu = 2, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
r = n - 2, \quad \mu = 3, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
r = n - 2, \quad \mu = 2, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
r = n - 1, \quad \mu = 4, \quad N_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

The Conditions of Non-Group-Membership.

18. The characteristic properties of a non-group-member M, any one of which distinguishes it from all group-members, may be summarized as follows:

(1) *M^2 is of lower rank than M;*

(2) *M is similar to a matrix having at least one nilpotent component;*

(3) *its characteristic determinant has at least one vacuous elementary divisor that is not linear;*

(4) *its vacuity is >1, and its rank is greater than its degree minus its vacuity;*

(5) *its group-index is >1.*

Non-Group-Members Whose Powers Are not All Distinct.

19. As a special case of a non-group-member, consider a matrix whose powers are not all distinct, while no higher power is equal to the first. Every such matrix M satisfies an equation of the form $M^{\mu+m} = M^{\mu}$ $(\mu > 1)$. If μ and m are the smallest positive integers for which this equation holds, than μ is evidently the group-index of M, while m is the order of the cyclic group formed by the higher powers of M.

For instance, if

$$M = \begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ then } M^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and M satisfies the equation $M^6 = M^2$; its group-index is two, and the higher powers M^2, M^3, M^4, M^5 form a cyclic group of order four, whose identical matrix is M^4.

In the canonical form (1) of a matrix of this kind the component L_u must be a periodic non-singular matrix of period m (if $u > 0$) and the component N_v must be a nilpotent matrix of group-index μ. That is, $L_u^m = U_u$ and $N_v^{\mu} = 0_v$. An immedite consequence is the

THEOREM: *A necessary and sufficient condition that the powers of a non-group-member are not all distinct is that the roots of its characteristic equation are either zero or roots of unity (at least two roots being zero) and that at least one of the vacuous elementary divisors of its characteristic determinant is non-linear, while all the non-vacuous elementary divisors are linear.**

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* This theorem, without its group implications, is due to Frobenius, "Ueber Lineare Substitutionen und Bilineare Formen," *Crelle's Journal*, Vol. LXXXIV (1878), p. 16, VIII.

Non-Group-Members Whose Higher Powers Are Equal.

20. A still more special case, included under that just considered, is obtained by putting $m=1$; the equation which M satisfies is then $M^{\mu+1}=M^{\mu}\ (\mu>1)$. The cyclic group generated by the higher powers of M is of order one, and M^{μ} is an idempotent matrix. The canonical form of M is

$$\begin{pmatrix} U_u & 0 \\ 0 & N_v \end{pmatrix}.$$

Hence the

THEOREM: *A necessary and sufficient condition that the higher powers of a non-group-member are all equal to one another is that the roots of its characteristic equation are equal to zero or one (at least two roots being zero) and that at least one of the vacuous elementary divisors of its characteristic determinant is non-linear, while all the non-vacuous elementary divisors are linear.*

It is clear that all nilpotent matrices are included in this class; for if M is a nilpotent matrix of degree n and group-index μ, it will satisfy the equation $M^{\mu}=M^{\mu+1}=0_n\ (\mu>1)$.

PART III. THE ROOTS OF A MATRIX.

Associated Matrices.

21. Two matrices M_1 and M_1' will be said to be *associated*, if there exists a non-singular matrix L that transforms them into

$$M=\begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \text{ and } M'=\begin{pmatrix} L_u & 0 \\ 0 & N_v' \end{pmatrix}, \tag{10}$$

respectively, where L_u is non-singular (if $u>0$), and N_v and N_v' are nilpotent or zero. If the vacuity of a matrix is zero or one, it has no associated matrices besides itself. On the other hand, if the vacuity of a matrix is >1, it has an infinite number of associates, which may be said to form a *set of associates.*

In every set of associates just one matrix, the one for which $N_v=0_v$, is a group-member, and all the rest are non-group-members. Thus every group-member of vacuity >1 has an infinite number of associated non-group-members.

Two sets of associates may be called similar, if there exists a non-singular matrix that transforms one into the other; the above set $\{M, M', \ldots\}$ may be called a canonical form of the similar set $\{M_1, M_1', \ldots\}$.

Sets of Associates.

22. THEOREM: *If two sets of associates S and S' have a matrix M' in common, they are identical.*

Proof. In the first place S and S' must obviously be similar sets; there will be no loss of generality in considering one of them, S, to be in its canonical form. The non-singular matrix L that transforms S' into S will transform the common matrix M' into some matrix M of S. Suppose M and M' to be given by equations (10) and let

$$L = \begin{pmatrix} A_u & B \\ C & D_v \end{pmatrix}.$$

Then since $M' L = L M$, we see that L must satisfy the equations

$$\text{(a) } L_u A_u = A_u L_u, \qquad \text{(b) } L_u B = B N_v,$$
$$\text{(c) } N_v' C = C L_u, \qquad \text{(d) } N_v' D_v = D_v N_v.$$

From (b) and (c), in view of a theorem due to Frobenius,[*] we see that $B = 0$, and $C = 0$, and therefore that

$$L = \begin{pmatrix} A_u & 0 \\ 0 & D_v \end{pmatrix};$$

from (a) we see that A_u is commutative with L_u, and therefore that L (and also L^{-1}) will transform the set S into itself. But L^{-1} transforms S back into S'; consequently the two sets S and S' are identical.

This theorem shows that every matrix belongs to one, and only one, set of associates, and that every non-group-member is associated with a uniquely determined group-member.

Pseudogroups.

23. The matrices of an entire group of vacuity > 1, together with all the non-group-members associated with them, constitute a set of matrices, which we shall call the *pseudogroup* associated with the given entire group. Hence a pseudogroup is made up of sets of associates. Since two different sets of associates can not have a matrix in common, and since the same is true of two different entire groups, therefore, similarly, two different pseudogroups can not have a matrix in common. In other words, pseudogroups are mutally exclusive sets of matrices. Accordingly, every matrix of vacuity > 1 belongs to one, and only one, pseudogroup, just as every matrix of vacuity one or zero belongs to one, and only one, entire group.

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[*] *Crelle's Journal*, Vol. LXXXIV (1878), p. 28, XI. The theorem is stated only for square matrices (bilinear forms), but can be immediately extended to cover the case in which B and C, as here, are rectangular matrices.

Since all entire groups of the same vacuity are similar, therefore all pseudogroups of the same vacuity are similar. The canonical form of every pseudogroup of vacuity v and degree $u+v$ may be written

$$P=\begin{pmatrix} G_u & 0 \\ 0 & H_v \end{pmatrix}, \tag{11}$$

where G_u (if $u>0$) denotes the group of non-singular matrices of degree u, and H_v denotes the totality of nilpotent matrices of degree v plus the matrix 0_v. The nilpotent matrices of degree n are all associated with the matrix zero, and with it form the only pseudogroup of vacuity n.

Since the group G_u, of (11), involves u^2 parameters, while in the set H_v the v^2 elements of each matrix are connected only by v independent relations derived from the vanishing of the roots of its characteristic equation, therefore *every pseudogroup of degree n and vacuity v is a continuous set of matrices depending on $(n-v)^2+v^2-v$ parameters.*

24. From the definition of a pseudogroup P it is clear that if a matrix M belongs to P, every power of M will belong to P; again, since pseudogroups are mutually exclusive, every root of M will belong to P. Hence the

THEOREM: *Every power and every root of a matrix of vacuity >1 belongs to the pseudogroup to which the matrix belongs; similarly, every power and every root of a matrix of vacuity one or zero belongs to the entire group to which the matrix belongs.*

On the other hand, the product of two matrices of a pseudogroup does not necessarily belong to the pseudogroup, because the product of two nilpotent matrices is not necessarily nilpotent.

The Roots of a Group-Member.

25. Let us now consider the distribution of the powers and roots of a matrix within the pseudogroup to which the matrix belongs. In the first place, if a pseudogroup P is of vacuity v, then the vth power of every one of its matrices belongs to the entire group with which P is associated.

Moreover, if, in formula (10) of § 21, the group-indices of M and M' are $\leqq \nu$, then

$$M^{\nu}=(M')^{\nu}=\begin{pmatrix} L_u^{\nu} & 0 \\ 0 & 0_v \end{pmatrix},$$

which we may call L. Accordingly, if we select from a set of associates those matrices whose group-indices are $\leqq \nu$, their νth powers must all be equal to one another and to the νth power, L, of their associated group-member. Therefore, *if M is a νth root of a group-member L, then, among the matrices associated with M, those whose group-indices are $\leqq \nu$ are all νth roots of L.*

In particular, if $\nu \geqq v$, then absolutely all the matrices associated with M are νth roots of L. That is, *the matrices of any set of associates of vacuity v are all νth roots of one and the same group-member, if $\nu \geqq v$.*

26. To find all the νth roots of any given group-member L of vacuity v, we can therefore proceed as follows: First find all the νth roots of L that are contained within the entire group G to which L belongs. Such roots exist for all positive integral values of the index ν. If v is one or zero, these are the only νth roots of L. But if $v > 1$, and if M is any νth root belonging to G, we then select those of its associated non-group-members, whose group-indices are $\leqq \nu$; in particular, if $\nu \geqq v$, we select all its associated non-group-members. The matrices so found will include all the νth roots of L and no other matrices.

It is evident, therefore, that every group-member L possesses roots of any given index ν.

The Roots of a Non-Group-Member.

27. Finally, let M' be a non-group-member of vacuity v and group-index μ belonging to the canonical pseudogroup P of (11), § 23, and let M be any νth root of M'; M will also belong to P. If we put

$$M = \begin{pmatrix} L_u & 0 \\ 0 & N_v \end{pmatrix} \text{ and } M' = \begin{pmatrix} L'_u & 0 \\ 0 & N'_v \end{pmatrix},$$

then M must satisfy the conditions $L_u^\nu = L'_u$ and $N_v^\nu = N'_v$. Now it is not always possible to find a nilpotent matrix N_v to satisfy the latter condition. But if it is possible, so that M' possesses a νth root M, then all its νth roots are connected as follows: Let L and L' be the group-members associated with M and M', respectively; then L is obviously a νth root of L'. Conversely, if L is a νth root of L', M is a νth root of M'. Hence the

THEOREM: *If M' is a non-group-member and L' its associated group-member, then if a νth root of M' exists, every νth root of M' is associated with a νth root of L' and every νth root of L' is associated with a νth root of M'.*

28. In order that M', as defined above, may possess a νth root, the inequality

$$(\mu - 1)\nu < v \qquad (12)$$

must be satisfied. For since μ is the group-index of M', the nilpotent matrix N'_v must satisfy the condition $(N'_v)^{\mu-1} \neq 0_v$; hence N_v must satisfy the condition

$(N_v)^{(\mu-1)\nu} \neq 0_v$. This means that the group-index of M is $> (\mu - 1)\nu$. But the maximum value of the group-index of M is its vacuity v, so that $v > (\mu - 1)\nu$. Therefore,

A non-group-member M' *of vacuity* v *and group-index* μ *does not possess a* ν*th root, unless* $(\mu - 1)\nu < v$. In other words, if $(\mu - 1)\nu \geqq v$, M' has no νth root, and, *a fortiori*, it has no root of index $> \nu$.

If in (12) v is fixed and μ has its minimum value 2, it follows that the maximum value of ν is $< v$. Hence, *a non-group-member can not have any roots of index equal to, or greater than, its vacuity.*

If in (12) v is fixed and ν has its minimum value 2, it follows that the maximum value of μ is $< \frac{v}{2} + 1$. Hence, *a non-group-member of vacuity* v *and group-index* $\geqq \frac{v}{2} + 1$ *has no square roots, and so has no roots whatever.*

Example. Consider ternary nilpotent matrices ($n = v = 3$), which have the two canonical forms:

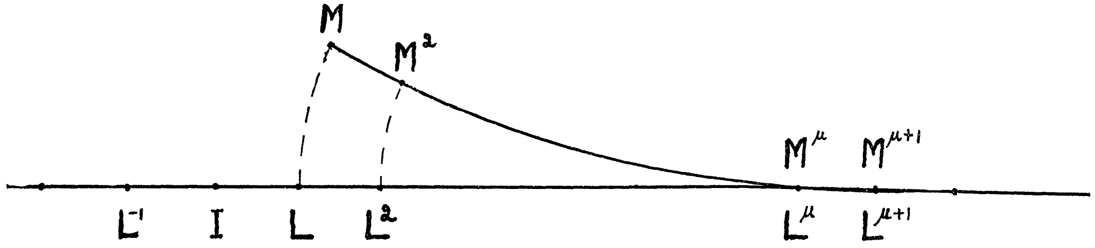
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\mu = 2), \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} (\mu = 3).$$

A has square roots, *e. g.* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, but no roots of higher index than two; on the other hand, B has no roots whatever.

PART IV. ILLUSTRATIONS AND APPLICATIONS.

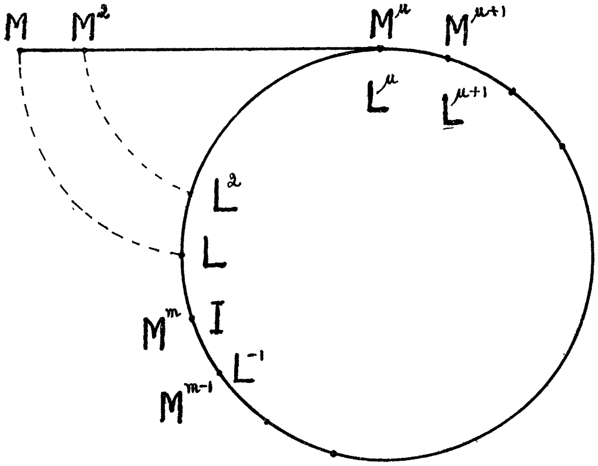
Graphical Representation.

29. In order to lend concreteness to the relation between the powers of a non-group-member M, of group-index μ, and the powers of its associated group-member L, we may represent them by points on two converging lines, thus:



The points on the straight line represent the matrices of the group G generated by L, in which I is the identical matrix and L^{-1} is the inverse of L. M is associated with L, M^2 with L^2, etc.; and since μ is the group-index of M, $M^{\mu} = L^{\mu}$ and all the higher powers of M are equal to the corresponding powers of L.* It is plain that although the powers of M from the μth on generate the group G, M^{μ} itself only generates a subgroup of G.

30. For the special case in which M satisfies the equation $M^{\mu+m} = M^{\mu}$, the matrices of the group G may be more conveniently represented by points on a circle, thus:



For the sake of simplicity m is here taken $> \mu$. As before, M is associated with L, M^2 with L^2, etc., and $M^{\mu} = L^{\mu}$, $M^{\mu+1} = L^{\mu+1}$, etc. Since L is of period m, the identical matrix of G is $I = L^m = M^m$. Moreover, $M^{\mu+m} = L^{\mu+m} = L^{\mu} = M^{\mu}$. The powers of M repeat themselves, but never reproduce the first power. M^{μ} will generate the whole group G, if, and only if, μ is prime to m. An instance in which it generates a subgroup is the example given in § 19.

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* By purely abstract considerations it is easy to show that if M is any mathematical entity whose μth and $(\mu+1)$st powers belong to a group G, then all its higher powers belong to G, provided the powers of M combine under the associative law. For since $(MM)M = M(MM)$, therefore M^{μ} and $M^{\mu+1}$ are commutative; and since the latter matrices belong to G, $(M^{\mu})^{-1}$ belongs to G and is commutative with $M^{\mu+1}$. Let $L = M^{\mu+1}(M^{\mu})^{-1} = (M^{\mu})^{-1}M^{\mu+1}$; then L belongs to G. Moreover, $L = M \cdot M^{\mu}(M^{\mu})^{-1} = (M_{\mu})^{-1}M^{\mu} \cdot M$. Therefore, if I is the identical element of G, $L = MI = IM$, and M is commutative with I. Hence $L^{\lambda} = M^{\lambda} I^{\lambda} = M^{\lambda} I$. Now suppose $\lambda = \mu + \nu$, where $\nu > 0$; then $L^{\lambda} = M^{\nu} \cdot M^{\mu} I = M^{\nu} \cdot M^{\mu}$ (since M^{μ} belongs to G) $= M^{\lambda}$; and since L belongs to G, M^{λ} belongs to G, when $\lambda \geqq \mu$. That is, every power of M, whose exponent is $\geqq \mu$, belongs to G.

Illustration from Differential Calculus.

31. The phenomenon of a mathematical entity not belonging to any group, while some of its powers do belong to a group, is not peculiar to the theory of matrices; it can also be exemplified elsewhere. For instance, in differential calculus, let D and E denote the operations of differentiating $y = e^{ax} + x^{\mu-1}$ and $z = e^{ax}$, respectively, as to x. Then

$$\begin{aligned} Dy &= a e^{ax} + (\mu - 1)\, x^{\mu-2} \text{ and } Ez = a e^{ax}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ D^{\mu} y &= a^{\mu} e^{ax} = E^{\mu} z, \\ &\ldots\ldots\ldots\ldots, \\ D^{\nu} y &= a^{\nu} e^{ax} = E^{\nu} z \qquad (\nu \geqq \mu). \end{aligned}$$

Now the powers of E form a group G, provided the inverse operation E^{-1} be defined by the equation $E^{-1} z = \frac{1}{a} e^{ax}$. On the other hand, although the powers of D evidently do not form a group, the higher powers of D, from the μth on, belong to the group G.

As a special case let a be a primitive mth root of unity; then G becomes a cyclic group of order m, and D an operation satisfying the equation

$$D^{\mu+m} y \; (= a^{\mu+m} e^{ax} = a^{\mu} e^{ax}) = D^{\mu} y.$$

Groups of Singular Collineations.

32. By the use of homogeneous point coordinates every matrix of degree n can be written as a linear substitution and interpreted geometrically as a collineation of a linear space, R_{n-1}, of $n-1$ dimensions. Every singular matrix will then carry the points of R_{n-1} into the points of an included space R_{r-1} $(r < n)$. Hence it is clear that in a group of singular collineations the identical collineation will not leave every point unchanged, as is the case in a group of non-singular collineations, and the inverse of a collineation that carries the point P into P' will not necessarily carry P' back into P.

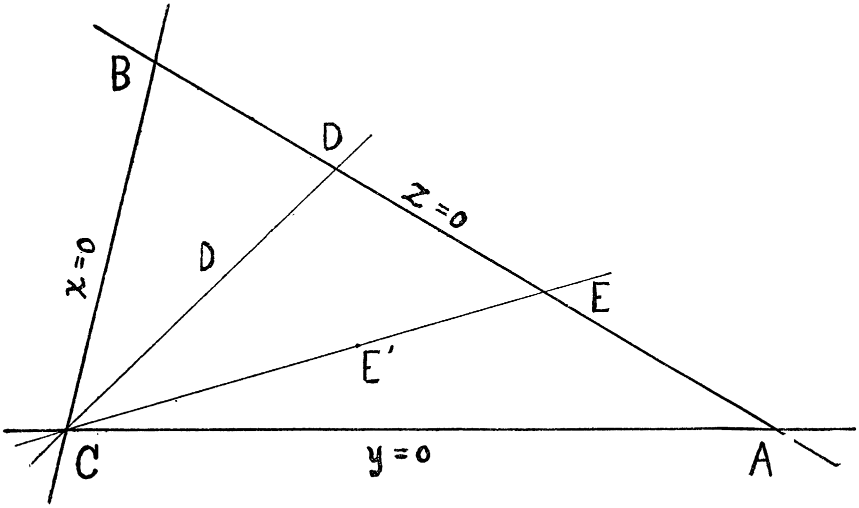
Nevertheless the group concept, as applied to singular collineations, can be justified geometrically, as will be apparent from the consideration of a simple case. Let n be 3 and consider a group generated by the singular matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of rank 2 and of infinite period. Then M, $M^0 = I$, and M^{-1} may be written as linear substitutions, thus:

$$M: \begin{cases} x' = x + y \\ y' = y \\ z' = 0 \end{cases} \qquad I: \begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases} \qquad M^{-1}: \begin{cases} x' = x - y \\ y' = y \\ z' = 0 \end{cases}$$

Considered as collineations in the plane they carry all the points of the plane except $C\ (x = y = 0)$ into points of the line $AB\ (z = 0)$.



All the points except C of any line passing through C are carried into one and the same point of AB, and the points of AB are transformed projectively among themselves. Let CD be any line passing through C, and suppose that M carries the points of CD into E; then M^{-1} will carry the points of CE into D. I evidently carries the points of CD into D, those of CE into E, etc.

Now the equations $M = IM = MI$ and $MM^{-1} = M^{-1}M = I$, which determine the group-membership of M, are satisfied geometrically. For I carries any point D' of the line CD into D, and M carries D into E; therefore IM carries D' into E, exactly as M does. Similarly MI transforms the points of the plane exactly as M does. Again, M^{-1} carries E' into D, and M carries D into E; therefore $M^{-1}M$ carries E' into E, just as I does. Similarly, MM^{-1} transforms the points of the plane just as I does.

This interpretation can obviously be extended to the general case of any group whatever of singular collineations.

Classification of Binary and Ternary Matrices.

33. An application of the principles derived in this paper will now be made to the classification of binary and ternary matrices, not only with respect to their group-membership, but also with respect to their vacuity v, rank r, group-index μ, and the entire groups and pseudogroups to which they belong. In these simple cases ($n = 2, 3$) the values of v and r completely determine the vacuous elementary divisors of the characteristic determinants of the matrices, and therefore also their characteristics,* so far as the vacuous elementary divisors alone are concerned.

34. *Binary Matrices.* Every binary matrix $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$ may be considered as belonging to one of four great classes, as follows:

$$\text{(a)} \quad v = 0, \ r = 2, \ \mu = 1; \ \alpha\delta - \beta\gamma \neq 0;$$

these matrices, ∞^4 in number, non-singular, form a single entire group.

$$\text{(b)} \quad v = 1, \ r = 1, \ \mu = 1; \ \alpha\delta - \beta\gamma = 0, \ \alpha + \delta \neq 0;$$

∞^3 matrices; characteristic $[1\,\overset{\circ}{1}]$; canonical form $\begin{pmatrix}\alpha & 0 \\ 0 & 0\end{pmatrix}$, $\alpha \neq 0$; they form ∞^2 similar entire groups, each containing ∞^1 matrices.

$$\text{(c)} \quad v = 2, \ r = 1, \ \mu = 2; \ \alpha\delta - \beta\gamma = \alpha + \delta = 0,$$

at least one element $\neq 0$; ∞^2 matrices; characteristic $[\overset{\circ}{2}]$; canonical form $\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$; they are nilpotent, all similar, and their square is zero; they are all associated with zero and with it form a single pseudogroup.

$$\text{(d)} \quad v = 2, \ r = 0, \ \mu = 1; \ \alpha = \beta = \gamma = \delta = 0;$$

characteristic $[(\overset{\circ}{1}\,\overset{\circ}{1})]$; one matrix, zero, forming an entire group of order one.

The matrices of classes (a), (b), and (d) are group-members, and those of class (c) are non-group-members. Thus all binary non-group-members are nilpotent. There are three kinds of binary idempotent matrices, viz., in class (a) the unit matrix $\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$, in class (b) the matrices for which $\alpha\delta - \beta\gamma = 0$ and $\alpha + \delta = 1$, ∞^2 in number, all similar, whose canonical form is $\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}$, and in class (d) the matrix zero.

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* See §2.

35. *Ternary Matrices.* Denoting a ternary matrix by the symbol (α_{ij}) $(i, j = 1, 2, 3)$, we shall define α'_{ij} as the cofactor of α_{ij} in the determinant $|\alpha_{ij}|$. Ternary matrices may be divided into seven classes, as follows:

$$\text{(a)}\ v = 0,\ r = 3,\ \mu = 1;\ |\alpha_{ij}| \neq 0;$$

∞^9 matrices, non-singular, forming a single entire group.

$$\text{(b)}\ v = 1,\ r = 2,\ \mu = 1;\ |\alpha_{ij}| = 0,\ \sum_1^3 \alpha'_{ii} \neq 0;$$

∞^8 matrices; canonical form $\begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha\delta - \beta\gamma \neq 0$; they are collected into ∞^4 similar entire groups, each containing ∞^4 matrices.

$$\text{(c)}\ v = 2,\ r = 2,\ \mu = 2;\ |\alpha_{ij}| = 0,\ \sum_1^3 \alpha'_{ii} = 0,\ \sum_1^3 \alpha_{ii} \neq 0,$$

at least one first minor $\alpha'_{ij} \neq 0$; ∞^7 matrices; characteristic $[1\,2]$; canonical form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \neq 0$; every matrix has ∞^6 similar matrices, of which ∞^2 are associated with it.

$$\text{(d)}\ v = 2,\ r = 1,\ \mu = 1;\ \alpha'_{ij} = 0\ \ (i, j = 1, 2, 3),$$

nine equations of condition, of which four are independent, $\sum_1^3 \alpha_{ii} \neq 0$; ∞^5 matrices; characteristic $[1(\overset{\circ}{1}\,\overset{\circ}{1})]$; canonical form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\alpha \neq 0$; they form ∞^4 similar entire groups, each containing ∞^1 matrices.

Classes (c) and (d) together consist of ∞^4 similar pseudogroups, the canonical form of which is the set of matrices of the form $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$, $\alpha \neq 0$, $ad - bc = a + d = 0$, so that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is nilpotent or zero. Each of these pseudogroups therefore contains ∞^3 matrices, of which ∞^1 are group-members of class (d) forming an entire group, and the rest are associated non-group-members of class (c); every group-member has ∞^2 associated non-group-members.

$$\text{(e)}\ v = 3,\ r = 2,\ \mu = 3;\ |\alpha_{ij}| = \sum_1^3 \alpha'_{ii} = \sum_1^3 \alpha_{ii} = 0,$$

at least one first minor $\alpha'_{ij} \neq 0$; ∞^6 matrices; characteristic $[\overset{\circ}{3}]$; canonical form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$; they are nilpotent, all similar, and their cube is zero.

$$\text{(f)}\ v = 3,\ r = 1,\ \mu = 2;\ \alpha'_{ij} = 0\ (i, j = 1, 2, 3),\ \sum_{1}^{3} \alpha_{ii} = 0,$$

at least one element $\alpha_{ij} \neq 0$; ∞^4 matrices; characteristic $[(\overset{\circ}{2}\,\overset{\circ}{1})]$; canonical form $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; they are nilpotent, all similar, and their square is zero.

$$\text{(g)}\ v = 3,\ r = 0,\ \mu = 1;\ \alpha_{ij} = 0\ (i, j = 1, 2, 3);$$

characteristic $[(\overset{\circ}{1}\,\overset{\circ}{1}\,\overset{\circ}{1})]$; one matrix, zero, forming an entire group of order one.

Classes (e), (f), and (g) together form a single pseudogroup, of which zero is the group-member, and all the other matrices are associated nilpotent non-group-members. The square of every matrix of (e) belongs to (f).

There are four kinds of ternary idempotent matrices, viz., in class (a) the unit matrix, in class (b) ∞^4 similar matrices whose canonical form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, in class (d) ∞^4 similar matrices whose canonical form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and in class (g) the matrix zero.

Application to Quaternions.

36. On account of the close connection that exists between the theory of matrices and the theory of hypercomplex numbers, it is clear that the concept of group-membership under multiplication can be transferred from the former to the latter. In particular, it is well-known that quaternions* are abstractly identical with binary matrices, both as to addition and multiplication.† The correspondence between them can be set up by making the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ correspond to the quaternion

$$\frac{\alpha + \delta}{2} - \frac{(\alpha - \delta)\sqrt{-1}}{2}\, i + \frac{\beta - \gamma}{2}\, j - \frac{(\beta + \gamma)\sqrt{-1}}{2}\, k$$

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* That is, quaternions of the general kind (sometimes called biquaternions), in which the coefficients of 1, i, j, k are ordinary complex, as well as real, numbers.

† This identification is due to Charles and Benjamin Peirce; see Taber, "On the Theory of Matrices," AMERICAN JOURNAL OF MATHEMATICS, Vol. XII (1890), p. 353.

for all values of α, β, γ, and δ, or, what is the same thing, by making the quaternion $a + bi + cj + dk$ correspond to the matrix

$$\begin{pmatrix} a + b\sqrt{-1} & c + d\sqrt{-1} \\ -c + d\sqrt{-1} & a - b\sqrt{-1} \end{pmatrix}$$

for all values of a, b, c, and d.

By means of this correspondence the classification of binary matrices in § 34 gives rise to a classification of quaternions. Thus every quaternion $a + bi + cj + dk$ belongs to one of four classes, as follows:

$$\text{(a)}\ a^2 + b^2 + c^2 + d^2 \neq 0;$$

∞^4 quaternions, forming a single group simply isomorphic with the entire group of non-singular binary matrices. Its identical element is the idempotent quaternion one.

$$\text{(b)}\ a^2 + b^2 + c^2 + d^2 = 0,\ a \neq 0;$$

∞^3 quaternions, whose canonical form is $\frac{\alpha}{2} - \frac{\alpha\sqrt{-1}}{2}i$ ($\alpha \neq 0$); they form a doubly infinite system of similar one-parameter groups. The identical elements of these groups are the idempotent quaternions defined by the equations $a^2 + b^2 + c^2 + d^2 = 0$, $a = \frac{1}{2}$; their canonical form is $\frac{1}{2} - \frac{\sqrt{-1}}{2}i$.

$$\text{(c)}\ a = 0,\ b^2 + c^2 + d^2 = 0,$$

at least one of the coefficients $b, c, d \neq 0$; ∞^2 quaternions, all similar; canonical form $\frac{1}{2}j - \frac{\sqrt{-1}}{2}k$; they are nilpotent, and their square is zero; together with zero they form a pseudogroup.

$$\text{(d)}\ a = b = c = d = 0;$$

the quaternion zero, forming a group of order one.

Therefore every quaternion either belongs to a multiplicative group or is nilpotent.

In a similar manner the results of this paper can be applied to other hypercomplex number-systems.

CORNELL UNIVERSITY, *March*, 1908.